

# Global Existence and Global Attractors of Cross Diffusion Systems on Planar Domains.

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## Abstract

Global existence of strong solutions and the existence of global and attractors are established for generalized Shigesada-Kawasaki-Teramoto models on planar domains. The cross diffusion and reaction can have polynomial growth of any order.

## 1 Introduction

Shigesada *et al.* in [17] introduce the following model

$$\begin{cases} u_t &= \operatorname{div}[\nabla(a_1u + \alpha_{11}u^2 + \alpha_{12}uv) + b_1u\nabla\Phi(x)] + f_1(u, v), \\ v_t &= \operatorname{div}[\nabla(a_2v + \alpha_{21}uv + \alpha_{22}v^2) + b_2v\nabla\Phi(x)] + f_2(u, v), \end{cases} \quad (1.1)$$

where  $f_i(u, v)$  are reaction terms of Lotka-Volterra type and quadratic in  $u, v$ . The unknowns  $u(x, t), v(x, t)$  denote the densities of two species at time  $t$  and location  $x \in \Omega$ , a bounded domain in  $\mathbb{R}^2$ . Dirichlet or Neumann boundary conditions were usually assumed for (1.1). This model was used to describe the population dynamics of the species  $u, v$  which move under the influence of population pressures and of the environmental potential  $\Phi(x)$ .

Under suitable assumptions of the coefficients in (1.1), Yagi proved in [19] the global existence of solutions to the above system for a planar domain  $\Omega$ . Clearly, (1.1) is a special case of the following system

$$u_t = \Delta(P(u)) + \hat{f}(u, Du) \quad (x, t) \in Q = \Omega \times (0, T), \quad (1.2)$$

where  $m \geq 2$ ,  $u : \Omega \rightarrow \mathbb{R}^m$ ,  $P : \mathbb{R}^m \rightarrow \mathbb{R}^m$ , whose components are *polynomials* in  $u$ , and  $\hat{f} : \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}^m$  are vector valued functions. The potential  $\Phi(x)$  is incorporated in  $\hat{f}(u, Du)$ , which will be assumed to have *linear* growth in  $Du$ . In this paper, we then refer to the above system as the *generalized (SKT) system* and allow  $P, \hat{f}$  to have *polynomial growth of any order* in  $u$ .

Under suitable assumptions of the parameters  $\alpha_{ij}$ 's in (1.1), Yagi proved in [19] that the solutions with positive initial data will stay positive and the Jacobian of  $P(u, v)$

$$A(u, v) = \begin{bmatrix} a_1 + 2\alpha_{11}u + \alpha_{12}v & \alpha_{12}u \\ \alpha_{21}v & a_2 + \alpha_{21}u + 2\alpha_{22}v \end{bmatrix}$$

is uniformly elliptic for  $u, v \geq 0$ . In fact, there are positive constants  $C$  and  $c_i$ 's, depending on the parameters  $d_i$ 's and  $\alpha_{ij}$ 's, and a  $C^1$  function  $\lambda(u, v) \sim c_0 + c_1u + c_2v$  such that for any  $\zeta \in \mathbb{R}^4$  and nonnegative  $u, v$  we have

$$\lambda(u, v)|\zeta|^2 \leq \langle A(u, v)\zeta, \zeta \rangle \text{ and } |A(u, v)| \leq C\lambda(u, v).$$

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In this paper, we consider the following generalized version of (1.2) of  $m$  equations ( $m \geq 2$ ) and rewrite it in a much more general form as

$$\begin{cases} u_t = \operatorname{div}(A(u)Du) + \hat{f}(u, Du) & (x, t) \in Q = \Omega \times (0, T), \\ u(x, 0) = U_0(x) & x \in \Omega \\ u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0, T). \end{cases} \quad (1.3)$$

We will always assume that the initial data  $U_0$  is given in  $W^{1,p_0}(\Omega, \mathbb{R}^m)$  for some  $p_0 > 2$ , the dimension of  $\Omega$ . As usual,  $W^{1,p}(\Omega, \mathbb{R}^m)$ ,  $p \geq 1$ , will denote the standard Sobolev spaces whose elements are vector valued functions  $u : \Omega \rightarrow \mathbb{R}^m$  with finite norm

$$\|u\|_{W^{1,p}(\Omega, \mathbb{R}^m)} = \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}.$$

Inspired by (1.1) and the above discussion, for  $m \geq 2$  we assume the following more general condition on the ellipticity of  $A(u)$ .

**A)**  $A(u)$  is  $C^1$  in  $u$  and there are constants  $\lambda_0, C > 0$  and a scalar  $C^1$  function  $\lambda(u)$  such that for all  $u \in \mathbb{R}^m$  and  $\zeta \in \mathbb{R}^{2m}$

$$\lambda(u) \geq \lambda_0, \quad \lambda(u)|\zeta|^2 \leq \langle A(u)\zeta, \zeta \rangle \text{ and } |A(u)| \leq C\lambda(u). \quad (1.4)$$

In addition,  $|A_u| \leq C|\lambda_u|$  and the following number is finite:

$$\mathbf{\Lambda} = \sup_{W \in \mathbb{R}^m} \frac{|\lambda_W(W)|}{\lambda(W)}. \quad (1.5)$$

Here and throughout this paper, if  $B$  is a  $C^1$  function in  $u \in \mathbb{R}^m$  then we abbreviate it derivative  $\frac{\partial B}{\partial u}$  by  $B_u$ .

Concerning the reaction term  $\hat{f}$ , we will assume the following.

**F)** There exist a constant  $C$  and a function  $f(u)$  such that

$$|\hat{f}(u, Du)| \leq C\lambda^{\frac{1}{2}}(u)|Du| + f(u), \quad (1.6)$$

$$|f_u(u)| \leq C\lambda(u). \quad (1.7)$$

Supposing that  $\lambda(u)$  is bounded from above, under the assumption A) and F), the global existence of (1.3) was studied in [1] for bounded domains  $\Omega \in \mathbb{R}^n$ ,  $n \geq 2$ . It was shown in [1] that a solution  $u$  of (1.3) exists globally if its  $W^{1,p}(\Omega)$  norm for some  $p > n$  does not blow up in finite time. First of all, the assumption that  $\lambda(u)$  is a constant or bounded does not apply to (1.3) in general because maximum principles are not available to show that  $u$  is bounded. Even if one knows that  $u$  is bounded, only its partial regularity properties is established, see [5].

In our recent work [12], estimates for the  $W^{1,p}(\Omega)$  norms for some  $p > n$  of a strong solution and then its global existence were established for (1.3) under A), F) and the weakest assumption that this solution is uniformly VMO (Vanishing Mean Oscillation) in the assumption M') of [12]. No boundedness assumption is needed. The proof in [12] relies

on fixed point theories, instead of the semigroup approach in [1], and weighted Gagliardo-Nirenberg inequalities involving BMO norms. However, the checking of the uniform VMO assumption in [12] is not a simple task.

In this paper, for planar domains  $\Omega \subset \mathbb{R}^2$  we will show in Theorem 2.1 that it is sufficient to control the  $W^{1,2}(\Omega)$  of a strong solution of (1.3) to establish its global existence. If this can be done for all initial data  $U_0$  in  $X = W^{1,p_0}(\Omega)$  then (1.3) defines a global semigroup  $\{\mathcal{S}(t)\}_{t \geq 0}$  on  $X$ , namely

$$\mathcal{S}(0)U_0 = U_0, \quad \mathcal{S}(t)U_0(x) = u(x, t)$$

is defined for all  $t > 0$ , with  $u$  being the solution of (1.3). We will show further in Theorem 2.3 that if the  $W^{1,2}(\Omega)$  norms of the strong solutions are uniformly bounded for  $t$  large then this semigroup possesses a global attractor and exponential attractors in  $X$ . Let us recall the definition of a global attractor: A set  $\mathcal{A} \subset X$  is a global (or universal) attractor if 1)  $\mathcal{A}$  is an invariant set ( $S(t)\mathcal{A} = \mathcal{A}$ ,  $\forall t \geq 0$ ), 2) For any  $U_0 \in X$   $S(t)U_0$  converges to  $\mathcal{A}$  as  $t \rightarrow \infty$ . in the Banach space  $X$ . The notion of exponential attractors in Hilbert spaces was introduced in [4] and the same definition applies for Banach spaces.

We state our main results in Section 2 and present their proof in Section 3. In Section 4, we apply our main theorems to the generalized SKT system (1.2). We will assume that  $\lambda(u)$  has polynomial growth in  $u$ , i.e.  $\lambda(u) \sim \lambda_0 + \lambda_1|u|^k$  for some  $k > 0$ , and the results in our main theorems continues to hold under much weaker assumptions. Namely, one needs only control the  $\|u\|_{L^q(\Omega)}$  for some  $q > k$ . In fact, for  $k \in (0, 2]$  it is sufficient to estimate  $\|u\|_{L^1(\Omega)}$  (the case  $k > 2$  needs a mild assumption on the number  $C_*$  in A)). We conclude the paper by showing that this is the case if the reaction term of (1.2) is of competitive type in some sense.

## 2 Preliminaries and Main Results

We state the main results of this paper in this section. Their proof will be given in the next section.

Our first result concerns the global existence of strong solutions to (1.3). We imbed (1.3) in the following family of systems

$$\begin{cases} u_t = \operatorname{div}(A(\sigma u)Du) + \hat{f}(\sigma u, \sigma Du) & (x, t) \in Q = \Omega \times (0, T_0), \sigma \in [0, 1] \\ u(x, 0) = U_0(x) & x \in \Omega \\ u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T_0). \end{cases} \quad (2.1)$$

In applications,  $\lambda(u)$  usually has polynomial growth in  $u$  so that we will introduce the following stronger version of (1.5).

$$|\lambda_u(u)| \leq \Lambda_1 \lambda^{1-\varepsilon_0}(u) \quad \text{for some } \Lambda_1, \varepsilon_0 > 0 \text{ and all } u \in \mathbb{R}^m. \quad (2.2)$$

**Theorem 2.1** *Let  $\Omega \subset \mathbb{R}^2$ . Suppose A), F), (2.2) and that we can establish the following: For any  $T_0 > 0$  there is a constant  $M(U_0, T_0)$  depending on  $\|U_0\|_{W^{1,p_0}(\Omega)}$  and  $T_0$  such that any strong solution  $u_\sigma$  of (2.1) on  $\Omega \times (0, T_0)$*

$$\sup_{t \in (0, T_0)} \|u_\sigma(\cdot, t)\|_{W^{1,2}(\Omega)} \leq M(U_0, T_0), \quad (2.3)$$

and for some  $s_0 > 0$

$$\sup_{t \in (0, T_0)} \|\lambda^{s_0}(u_\sigma(\cdot, t))\|_{L^1(\Omega)} \leq M(U_0, T_0). \quad (2.4)$$

Then (1.3) has a unique solution which exists globally.

**Remark 2.2** The assumptions on the bounds for solutions of (2.1) in Theorem 2.3 are not very restrictive as they seem at first glance. In applications, since the systems in (2.1) usually satisfy A) and F) for the same set of constants so that we need to check the estimates (2.3) and (2.4) only for  $\sigma = 1$ . Furthermore, the reaction terms  $\hat{f}(\sigma u, \sigma Du)$  can also be replaced by  $\hat{f}_\sigma(u, Du)$  if these functions satisfy F) uniformly.

Thus, if (2.3) holds for any initial data  $U_0 \in X := W^{1,p_0}(\Omega)$  then (1.3) defines a global semigroup  $\{\mathcal{S}(t)\}_{t \geq 0}$  on  $X$ , namely

$$\mathcal{S}(0)U_0 = U_0, \quad \mathcal{S}(t)U_0(x) = u(x, t)$$

is defined for all  $t > 0$ , with  $u$  being the solution of (1.3).

Next, we will show that if the norm  $\|u(\cdot, t)\|_{W^{1,2}(\Omega)}$  can be bounded uniformly for bounded initial data  $U_0 \in X$  when  $t$  is sufficiently large then the global dynamical systems  $\{\mathcal{S}(t)\}_{t \geq 0}$  possesses a global attractor in  $X$ .

**Theorem 2.3** *Assume as in Theorem 2.1. Suppose further that the dynamical system defined by (1.3) possesses an absorbing ball in  $W^{1,2}(\Omega)$ . That is there is a constant  $M$  such that for any bounded set  $K \subset W^{1,p_0}(\Omega)$  there is  $T_K > 0$  such that any global solution  $u$  of (1.3) with initial data  $U_0 \in K$  will satisfy*

$$\|u(\cdot, t)\|_{W^{1,2}(\Omega)} \leq M \quad \text{for all } t \geq T_K, \quad (2.5)$$

and for some  $s_0 > 0$

$$\|\lambda^{s_0}(u(\cdot, t))\|_{L^1(\Omega)} < M \quad \text{for all } t \geq T_K. \quad (2.6)$$

Then the system (1.3) possesses a global attractor in  $X = W^{1,p_0}(\Omega)$ .

We remark that the conditions in the above theorem also show that  $\{\mathcal{S}(t)\}_{t \geq 0}$  possesses exponential attractors in the Banach space  $X$ . The notion of exponential attractors in Hilbert spaces was introduced in [4]: A set  $\mathcal{A} \subset X$  is an exponential attractor if 1)  $\mathcal{A}$  is an positively invariant set ( $\mathcal{S}(t)\mathcal{A} \subset \mathcal{A}$ ,  $\forall t \geq 0$ ), 2) For any  $U_0 \in X$   $\mathcal{S}(t)U_0$  converges exponentially to  $\mathcal{A}$  as  $t \rightarrow \infty$ . The same definition applies when  $X$  is a Banach space. It is shown in [11] that this notion is quite universal: if  $\{\mathcal{S}(t)\}_{t \geq 0}$  possesses a global attractor and  $\mathcal{S}(t)$  is a  $C^1$  compact map on  $X$  for all  $t > 0$  then exponential attractors exist. In the next section, where we present the proof of the above theorems, higher regularity of  $u$  will be established and we will see that  $Du$  is Hölder continuous in  $(x, t)$  (see (3.9)). From this, we can prove that the  $\mathcal{S}(t)$  is  $C^1$  on  $X$  and thus, by [11],  $\{\mathcal{S}(t)\}_{t \geq 0}$  possesses exponential attractors in the Banach space  $X$ .

### 3 Global existence results in the general case and the proof of the main theorems

The proof of our main theorems relies on the global existence result in our recent work [12], which deals with the general case  $n \geq 2$ . In order to describe the main result in [12], we first recall the following main technical result and introduce the key condition M').

For the general case  $n \geq 2$ , we also assumed in [12] that

**SG)** (The spectral gap condition)  $(n-2)/n < C_*^{-1}$ .

**Lemma 3.1** *We assume that  $A, \hat{f}$  satisfy A), F) and SG). Suppose that  $u$  is strong solution of (1.3) on  $\Omega \times (0, T_0)$  and there are a constant  $C_{1,2}$ , which may depend on  $T_0$ , and a sufficiently large  $r$  such that*

$$\int_0^{T_0} \int_{\Omega} |Du(x, s)|^2 dx ds < C_{1,2}, \quad (3.1)$$

$$\|\lambda(u(\cdot, t))\|_{L^r(\Omega)} < C_{1,2}. \quad (3.2)$$

More importantly, we assume that

**M')** for any given  $\mu_0 > 0$  there is a positive  $R_{\mu_0}$  such that

$$\mathbf{\Lambda}^2 \sup_{x_0 \in \bar{\Omega}, t \in (0, T_0)} \|u(\cdot, t)\|_{BMO(B_{R_{\mu_0}}(x_0) \cap \Omega)}^2 \leq \mu_0. \quad (3.3)$$

Then if  $\mu_0$  is sufficiently small in terms of the parameters in A) then there are a number  $p > n$  and a constant  $C_{1,p}$  depending on the parameters in A), F) and  $\mathbf{\Lambda}, \mu_0, R_{\mu_0}$  and the geometry of  $\Omega$  such that

$$\|u(\cdot, t)\|_{W^{1,p}(\Omega)} \leq C_{1,p} \quad \forall t \in (0, T_0). \quad (3.4)$$

This result is a consequence of [12, Proposition 3.1] by taking  $\beta(u) = \lambda^{-1}(u)$  and  $W = U = u$ . The conditions required by this proposition that  $\|\beta(u(\cdot, t))\|_{L^r(\Omega)}$  is bounded and that  $\lambda(u)\beta(u)$  is a  $A_{\frac{4}{3}}$  weight are satisfied here because  $\lambda(u)$  is bounded from below and  $\lambda(u)\beta(u) = 1$ .

**Remark 3.2** The dependence of  $C_{1,p}$  in (3.4) on the geometry of  $\Omega$  means:  $C_{1,p}$  depends on a number  $N_{\mu_0}$  of balls  $B_{R_{\mu_0}}(x_i)$ ,  $x_i \in \bar{\Omega}$  and  $R_{\mu_0}$  is as in M'), such that

$$\bar{\Omega} \subset \cup_{i=1}^{N_{\mu_0}} B_{R_{\mu_0}}(x_i). \quad (3.5)$$

The bound (3.4) was established locally for balls  $B_{R_{\mu_0}}(x_i) \subset \Omega$  with  $R_{\mu_0}$  satisfying M'). If  $x_i$  is on the boundary  $\partial\Omega$  then a flattening and odd/even reflection arguments, depending on the type of boundary condition of  $u$ , can apply to extend the proof for the interior case to the boundary one. Adding these estimates, we obtain the global (3.4). Hence,  $N_{\mu_0}$  and  $C_{1,p}$  also depend on the geometry of  $\partial\Omega$ . Also, from the proof one can see that  $C_{1,p}$  also depends on  $R_{\mu_0}^{-1}$ .

Lemma 3.1 is the key ingredient in the proof of the following theorem, which is a consequence of [12, Theorem 4.2] by taking  $\beta(u) = \lambda^{-1}(u)$  again.

**Theorem 3.3** *Suppose that (1.3) satisfies A), F). We consider the following family of systems*

$$\begin{cases} u_t = \operatorname{div}(A(\sigma u)Du) + \hat{f}(\sigma u, \sigma Du) & (x, t) \in Q = \Omega \times (0, T_0), \sigma \in [0, 1] \\ u(x, 0) = U_0(x) & x \in \Omega \\ u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T_0). \end{cases} \quad (3.6)$$

*Assume that the strong solutions of these systems satisfy (3.1), (3.2) and M') uniformly. Then (1.3) has a unique strong solution  $u$  which exists globally on  $\Omega \times (0, \infty)$ .*

There is a subtle point in the assumptions in [12, Theorem 4.2] that is worth discussing here. The paper [12] dealt with the general case where  $n \geq 2$ ,  $A, \hat{f}$  depend also on  $x, t$  and so does  $\lambda$ . It then assumed that  $\lambda(x, t, u)$  is bounded near  $t = 0$ . It turns out that this is not needed for the global existence result. In this paper we assume that  $A, \hat{f}$  are independent of  $x, t$  so that we need to explain this matter further by sketching the main ideas in proof of [12, Theorem 4.2] below.

The proof of [12, Theorem 4.2], or its special case Theorem 3.3 here, makes use of fixed point theorems in a very standard way by considering the linear compact maps associated to the systems (3.6) on the Banach space  $\mathcal{X} = \mathcal{X}_1 \cap \mathcal{X}_3$  (the space  $\mathcal{X}_2 = C(\Omega \times (0, t_0))$  in the proof of [12, Theorem 4.2] is not needed here) where

$$\mathcal{X}_1 = C((0, T_0), C^{\alpha_0}(\Omega)) \text{ and } \mathcal{X}_3 = \{u : Du \in C^{\alpha_0, \alpha_0/2}(\Omega \times (t_0, T_0))\}.$$

Here,  $t_0 \in (0, T_0)$  is fixed and  $\alpha_0 > 0$  is a number such that  $W^{1,p}(\Omega) \cap W^{1,p_0}(\Omega)$ , with  $p > n$  being given in (3.4), is compactly imbedded in  $C^{\alpha_0}(\Omega)$ . We define  $\|u\|_{\mathcal{X}} = \|u\|_{\mathcal{X}_1} + \|u\|_{\mathcal{X}_3}$ , where

$$\|u\|_{\mathcal{X}_1} = \sup_{t \in (0, T_0)} \|u\|_{C^{\alpha_0}(\Omega)} \text{ and } \|u\|_{\mathcal{X}_3} = \sup_{t \in (t_0, T_0)} \|Du\|_{C^{\alpha_0, \alpha_0/2}(\Omega)}.$$

For each  $w \in \mathcal{X}$  and  $\sigma \in [0, 1]$  we define  $u = T_{\sigma}(w)$  to be the weak solution of

$$\begin{cases} u_t = \operatorname{div}(A(\sigma w)Du) + \hat{f}(\sigma w, \sigma Dw) & (x, t) \in Q = \Omega \times (0, T_0), \sigma \in [0, 1] \\ u(x, 0) = U_0(x) & x \in \Omega \\ u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T_0). \end{cases} \quad (3.7)$$

The strong solution of (1.3) on the cylinder  $Q$  is then the fixed point of the corresponding map,  $\sigma = 1$ , in  $\mathcal{X}$ .

From the regularity theory of linear systems with smooth data (see [14, Chapter 4] or [3]), we see that the solution of the above system is in  $\mathcal{X}$ . Furthermore, the higher regularity estimates in [3, 14] also show that  $T_{\sigma}$  is a compact map on  $\mathcal{X}$ .

We now discuss the uniform boundedness of the fixed points  $u = T_{\sigma}(u)$ . First of all, we observe that  $u$  is a strong solution on  $(0, T_0)$ . In fact, as  $u \in \mathcal{X}$  we have that  $Du$  is Hölder continuous in  $Q$ . Using the regularity theory of linear systems with smooth data again, we see that  $u$  is a strong solution in  $\Omega \times I$ . Therefore, Lemma 3.1 is applicable here.

The argument in [12, Proposition 3.1], which gives Lemma 3.1, made use of a cutoff function  $\eta$  for the interval  $[0, T + t_0]$  and  $[0, T + 2t_0]$ , that is  $\eta(t) = 0$  for  $t < T + t_0$  and  $\eta(t) = 1$  for  $t > T + 2t_0$ , to avoid the dependence on the initial data at  $t = 0$ . If we allow  $t_0 = 0$  then the bound for  $\|u(\cdot, t)\|_{W^{1,p}(\Omega)}$  in [12, Proposition 3.1] (see also [12, inequality 3.13]) is independent of  $t_0$  but  $\|u(\cdot, 0)\|_{W^{1,p}(\Omega)}$ . Of course, we can take  $p \in (n, p_0]$  so that (3.4) holds for all  $t \in [0, T_0]$  with  $C_{1,p}$  explicitly depending on  $\|U_0\|_{W^{1,p_0}(\Omega)}$ . We then obtain a uniform bound for  $\|u(\cdot, t)\|_{W^{1,p}(\Omega)}$ ,  $t \in (0, T_0)$ . Hence,  $\|u\|_{\mathcal{X}_1}$  is uniformly bounded for any fixed points  $u$  of  $T_\sigma$ .

Now, for any strong solution  $u$  to (3.6) and cylinder  $Q_R = B_R \times (t - R^2, t)$  in  $Q$  we can use Hölder's inequality (in the  $x$  integral) and (3.4) to obtain ( $dz = dxdt$ )

$$R^{-n} \iint_{Q_R} |Du|^2 dz \leq R^{-n+2+n(1-\frac{2}{p})} C_{1,p}^{\frac{2}{p}} = R^{2(1-\frac{n}{p})} C_{1,p}^{\frac{2}{p}}. \quad (3.8)$$

Since  $p > n$ , for any given  $\varepsilon_0 > 0$  there is  $R_0 = R_0(\varepsilon_0, C_{1,p}) > 0$  such that

$$\sup_{R \leq R_0} R^{-n} \iint_{Q_R} |Du|^2 dz < \varepsilon_0.$$

Since  $p > 2$ ,  $u$  is bounded by (3.4) and Sobolev's embedding theorems so that  $\lambda(u)$  is uniformly bounded, we can apply [5, Theorem 3.1] to see that  $u \in C^{\alpha, \alpha/2}(Q)$  for all  $\alpha \in (0, 1)$ . As  $A(\sigma u)$  is  $C^1$  in  $u$ ,  $A(\sigma u(x, t)) \in C^{\alpha, \alpha/2}(Q)$  so that [5, Theorem 3.2] applies to show that  $Du \in C_{loc}^{\alpha, \alpha/2}(Q)$  for all  $\alpha \in (0, 1)$ . In fact, it was shown in [5] that there are  $R_* > 0$  and a constant  $C$  depending on the parameters in A), F) and  $C_{1,p}$  such that

$$\|Du\|_{C^{\alpha, \alpha/2}(B_R \times (t - R^2, t))} \leq C \text{ if } R \leq R_* \text{ and } B_{2R} \times (t - 4R^2, t) \subset Q. \quad (3.9)$$

The above inequality also holds if the center of  $B_R$  is on the boundary  $\partial\Omega$  so that with  $4R^2 < t_0$  and  $t - 4R^2 > 0$  we see that  $\|u\|_{\mathcal{X}_3}$  is uniformly bounded.

Thus, we obtain a uniform bound for the fixed points of  $T_\sigma$  in the Banach space  $\mathcal{X}$  and the Leray-Schauder theorem can apply to give the existence of a strong solution in  $Q$  for any given  $T_0 > 0$ . By the uniqueness of strong solutions (as  $u$  is bounded and  $A, \tilde{f}$  are smooth) we see that the strong solutions in two cylinders  $Q \subset Q'$  coincide in  $Q$ . This shows that  $u$  is unique and exists globally. We now see that the proof continues to hold without the boundedness assumption of  $\lambda(x, t, u)$  for  $t$  near 0 in [12].

Since the systems (3.6) satisfy A), F) and SG) with the same set of constants so that, in applications, we need only to verify (3.1), (3.2) and M') for strong solutions to (1.3) then the same argument and Lemma 3.1 shows that the estimate (3.4) also holds uniformly for the strong solutions of the systems in the family.

The assumption (3.1) is usually easy to check by testing the systems with  $u$ . Meanwhile, (3.2) is also a mild assumption, especially if  $\lambda(u)$  has polynomial growth as in (1.1) and we know that  $u$  is BMO (see also (2.4) and Lemma 3.4). Thus, M') is the key assumption needs to be checked in order to establish (3.4) and then the bound (3.9) for higher norms of the solution  $u$ . The numbers  $\mu_0, R_{\mu_0}$  are the key parameters determining  $C_{1,p}$  and these bounds. One should note that the number  $R_{\mu_0}$  may also depend on the initial condition  $u(\cdot, 0)$  so that the bound  $C_{1,p}$  may implicitly depend on  $\|u(\cdot, 0)\|_{W^{1,p}(\Omega)}$ .

We are now ready to give the proof of Theorem 2.1 for the planar case  $\Omega \subset \mathbb{R}^2$  by checking the conditions of Theorem 3.3 under the assumption (2.3) that the norm  $\|u(\cdot, t)\|_{W^{1,2}(\Omega)}$  of any strong solution  $u$  of (1.3) does not blow up in finite time.

First of all, we observe that the assumption (2.4) in Theorem 2.1 is much weaker than (3.2) in Lemma 3.1, which requires higher integrability of  $\lambda(u)$ . This is because we have assumed (2.3) which is a bit stronger than M'). We have the following lemma showing that (2.2), (2.3) and (2.4) imply (3.2). This lemma will be also used in the proof of Theorem 2.3.

**Lemma 3.4** *Fix a  $t \in (0, T_0)$ . Assume (2.2) and that there are positives  $s_0, M_0, M_1$  and  $C_0$  such that*

$$\|\lambda^{s_0}(u(\cdot, t))\|_{L^1(\Omega)} \leq C_0 M_0^{s_0}, \quad \|Du(\cdot, t)\|_{L^2(\Omega)} \leq M_1. \quad (3.10)$$

*Then, for any  $r > 1$  there is a constant  $C(C_0, s_0, r, |\Omega|, \mathbf{\Lambda}_1, M_0, M_1)$  such that*

$$\|\lambda(u(\cdot, t))\|_{L^r(\Omega)} \leq C(C_0, s_0, r, |\Omega|, \mathbf{\Lambda}_1, M_0, M_1). \quad (3.11)$$

**Proof:** We choose and fix  $s_* > 0$  and  $p \in (1, 2)$  such that  $s_* p_* = s_0$ , where  $p_* = 2p/(n-2)$ . Then (3.10) implies

$$\|\lambda^s(u)\|_{L^{p_*}(\Omega)} \leq C_0 \lambda_*, \quad \text{where } s = s_* p_* \text{ and } \lambda_* := M_0^{\frac{s_0}{p_*}}. \quad (3.12)$$

Fix a  $t \in (0, T_0)$  and define  $g(\cdot) = \lambda^{s+\varepsilon_0}(u(\cdot, t))$ . The definition of  $\mathbf{\Lambda}_1$  in (2.2) gives

$$|Dg| \leq C(s) \frac{|\lambda_u|}{\lambda^{1-\varepsilon_0}(u)} \lambda^s(u) |Du| \leq C(s) \mathbf{\Lambda}_1 \lambda^s(u) |Du|.$$

Hence, as  $q = (2/p)'$ , by Hölder's inequality,  $\|Dg\|_{L^p(\Omega)} \leq C \|\lambda^s(u)\|_{L^{p_*}(\Omega)} \|Du\|_{L^2(\Omega)}$ . This implies, using (3.12) and (3.10),  $\|g\|_{W^{1,p}(\Omega)} \leq C(C_0, s, \mathbf{\Lambda}_1, M_1) \lambda_*$ . By Sobolev's imbedding theorem,

$$\|\lambda^{s+\varepsilon_0}(u)\|_{L^{p_*}(\Omega)} = \|g\|_{L^{p_*}(\Omega)} \leq C(C_0, s, \mathbf{\Lambda}, M_1) \lambda_*.$$

This shows that if (3.12) holds for some  $s$  then it also holds for  $s$  being  $s + \varepsilon_0$  and new constants  $C_1$  depending on  $M_0, M_1$ . It is clear that we can repeat this argument to see that  $\|\lambda^{s+k\varepsilon_0}(u)\|_{L^{p_*}(\Omega)} \leq C_k \lambda_*$  for all integers  $k$ . A simple use of Hölder's inequality and the definition of  $\lambda_*$  complete the proof. ■

**The proof of Theorem 2.1** We will make use of Lemma 3.1 and check its assumptions here. First of all, because we are considering the case  $n = 2$  the condition SG) is trivially satisfied. Next, thanks to Lemma 3.4, it is now clear that the condition (3.2) holds under its weaker version (2.4) and (2.3).

Therefore, we only need to show that M') holds for (2.1). The argument after Theorem 3.3, whose (3.6) is exactly (2.1), then shows that the strong solution  $u$  of (1.3) exists globally. Since  $\mathbf{\Lambda}$  is bounded, we need only prove that for some sufficiently small  $R$  and any ball  $B_R$  and any  $t$  in  $(0, T_0)$  the quantity  $\|u_\sigma(\cdot, t)\|_{BMO(B_R \cap \Omega)}$  can be arbitrarily small. We argue by contradiction. If this is not the case then there are sequences  $\{x_n\} \subset \bar{\Omega}$ ,  $\{\sigma_n\} \subset [0, 1]$ ,  $\{t_n\} \subset (0, T_0)$ ,  $\{r_n\}$ ,  $r_n \rightarrow 0$ , such that

$$\|u_{\sigma_n}(\cdot, t_n)\|_{BMO(B_{r_n}(x_n) \cap \Omega)} > \varepsilon_0 \text{ for some } \varepsilon_0 > 0.$$



Let  $U_n(\cdot) = u_{\sigma_n}(\cdot, t_n)$ . By (2.3) we see that the sequence  $\{U_n\}$  is bounded in  $W^{1,2}(\Omega)$ . We can then assume that  $U_n$  converges weakly to some  $U$  in  $W^{1,2}(\Omega)$  and strongly in  $L^2(\Omega)$ . We then have  $\|U_n\|_{BMO(B_R \cap \Omega)} \rightarrow \|U\|_{BMO(B_R \cap \Omega)}$  for any given ball  $B_R$ . Since  $n = 2$ , by Poincaré's inequality  $U$  is VMO and  $\|U\|_{BMO(B_R \cap \Omega)} < \varepsilon_0/2$  if  $R$  is sufficiently small. Furthermore, we can assume also that  $x_n$  converges to some  $x \in \bar{\Omega}$ . Thus, for large  $n$ , we have  $r_n < R/2$  and  $x_n \in B_{R/2}(x)$ . Hence,  $B_{r_n}(x_n) \subset B_R(x)$  and if  $n$  is sufficiently large then

$$\|U_n\|_{BMO(B_{r_n}(x_n) \cap \Omega)} \leq \|U_n\|_{BMO(B_R(x) \cap \Omega)} \leq \|U\|_{BMO(B_R(x) \cap \Omega)} + \varepsilon_0/2 < \varepsilon_0.$$

We obtain a contradiction and complete the proof. ■

We now give the proof of Theorem 2.3.

**Proof of Theorem 2.3** From the theory of global attractors (e.g. see [18]) it is now well known that we need only establish the following claims:

**Claim 1:** (1.3) defines a global semigroup  $\{\mathcal{S}(t)\}_{t \geq 0}$  in the Banach space  $X = W^{1,p_0}(\Omega)$ . Namely, the maps

$$\mathcal{S}(0)U_0 = U_0, \quad \mathcal{S}(t)U_0(x) = u(x, t)$$

with  $u$  being the solution of (1.3) for the given initial data  $U_0 \in X$ , is defined for all  $t > 0$ .

**Claim 2:** The map  $\mathcal{S}(t)$  is compact and possesses an absorbing ball in  $X$ .

As we assume (2.5) holds for any initial data  $U_0 \in X$ , Claim 1 is already established in Theorem 2.1 proving the global existence of (1.3).

We need only consider Claim 2. The discussion after Theorem 3.3 on the regularity of the solutions shows that  $\mathcal{S}(t)$  is a compact map on  $X$ . Moreover, (3.9) shows that the norm  $\|u(\cdot, t)\|_{C^1(\Omega)}$ , and thus  $\|u(\cdot, t)\|_X$ , is bounded in terms of the constant  $C_{1,p}$  in (3.4) of Lemma 3.1. We should note that the cutoff function  $\eta(t)$  can now be used for some fixed  $t_0$  so that the bound  $C_{1,p}$  is independent of the initial data  $u(\cdot, 0)$  but  $t_0^{-1}$ . Since  $u$  exists globally, we can choose  $t_0 = 1$ . Therefore, for any bounded set  $K \subset X$  if we can show that the assumption M') is verified uniformly for all  $u \in K$  and  $t$  sufficiently large then we can choose a universal  $C_{1,p}$  and then show that there is an absorbing ball in  $X$ .

Thus, we just need to show that the conditions of Lemma 3.1 hold uniformly for large  $t$  to give a uniform bound  $C_{1,p}$ . First of all, for large  $t$  in Lemma 3.4 we let  $M_0 = M^{1/s_0}$  and  $M_1 = M$ , where  $M$  is the universal constant in (2.5) and (2.6). By Lemma 3.4, the condition (3.2) of Lemma 3.1 holds uniformly when  $t$  is large.

To prove that M') holds uniformly, we use a contradiction argument similar to that in the proof of Theorem 2.1. We need only show that for some sufficiently small  $R$  and any ball  $B_R$  and any bounded set  $K$  the quantity  $\|u(\cdot, t)\|_{BMO(B_R \cap \Omega)}$ , where  $u$  is a strong solution with initial data  $U_0$  in  $K$ , can be arbitrarily small when  $t$  is sufficiently large. If this is not the case then there are sequences  $\{K_n\}$  of bounded sets in  $X$ ,  $\{u_n\}$  of strong solutions to (1.3) with initial data in  $K_n$ ,  $\{x_n\} \subset \bar{\Omega}$ ,  $\{t_n\} \subset (0, \infty)$  with  $t_n > T_{K_n}$ ,  $\{r_n\}$ ,  $r_n \rightarrow 0$ , such that

$$\|u_n(\cdot, t_n)\|_{BMO(B_{r_n}(x_n) \cap \Omega)} > \varepsilon_0 \text{ for some } \varepsilon_0 > 0.$$

Let  $U_n(\cdot) = u_n(\cdot, t_n)$ . By the same argument in the end of the proof of Theorem 2.1 we obtain a contradiction and complete the proof. ■

## 4 The Generalized SKT systems on Planar Domains

In this section, we consider the generalized version of the SKT system (1.1)

$$u_t = \Delta(P(u)) + \hat{f}(u, Du) \quad (x, t) \in Q = \Omega \times (0, T), \quad (4.1)$$

where  $u : \Omega \rightarrow \mathbb{R}^m$ ,  $P : \mathbb{R}^m \rightarrow \mathbb{R}^m$  and  $f : \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}^m$  are vector valued functions. It is clear that (4.1) generalizes (1.1), where  $P(u)$  is quadratic in  $u$ , but is still a special case of (1.3) for  $A(u) = P_u(u)$

$$u_t = \operatorname{div}(A(u)Du) + \hat{f}(u, Du) \quad (x, t) \in Q = \Omega \times (0, T_0).$$

Inspired by the usual SKT system (1.1), we will allow the following polynomial growth condition of  $A(u), \hat{f}(u)$ .

**SKT)** Assume that  $P(0) = 0$  and  $A(u) := P_u(u)$  satisfies A). Moreover, there are positive constants  $\lambda_0, \lambda_1, k, C$  and  $r_0$  such that for all  $u \in \mathbb{R}^m$

$$\lambda(u) \sim \lambda_0 + \lambda_1|u|^k, \quad |\lambda_u(u)| \leq \begin{cases} \text{bounded by } C\lambda_1|u|^{k-1} & \text{if } |u| > r_0, \\ C & \text{if } |u| \leq r_0. \end{cases} \quad (4.2)$$

In addition,  $|P(u)| \leq C\lambda(u)|u|$ .

Here and in the sequel, we will write  $a \sim b$  if there are two generic positive constants  $C_1, C_2$  such that  $C_1b \leq a \leq C_2b$ .

Firstly, it is easy to check that the assumption (1.5) in A) on the finiteness of

$$\Lambda = \sup_{u \in \mathbb{R}^m} \frac{|\lambda_u(u)|}{\lambda(u)}$$

is satisfied under the above polynomial growth assumption (4.2) on  $\lambda, \lambda_u$ .

Secondly, since  $P(0) = 0$  and  $A(u) = P_u(u)$  with  $|A(u)| \sim \lambda(u)$ , it is natural to assume  $|P(u)| \leq C\lambda(u)|u|$  for some constant  $C$ .

We also note that  $\lambda(u)$  is the smallest eigenvalue of  $(A + A^T)/2$  and  $\lambda^2(u)$  is the smallest eigenvalue of  $A^T A$ . Therefore, (1.4) implies

$$C\lambda^2(u)|\zeta|^2 \geq |A(u)\zeta|^2 = \langle A^T(u)A(u)\zeta, \zeta \rangle \geq \lambda^2(u)|\zeta|^2. \quad (4.3)$$

Similarly, we will assume the following assumption on the reaction terms.

**F')**  $\hat{f}$  satisfies F) and there is  $C_f > 0$  such that for  $\lambda_S = \lambda_0 + \lambda_1$

$$|f(u)| \leq C_f \lambda_S^{-1} |u| \lambda(u) \quad \forall u \in \mathbb{R}^m. \quad (4.4)$$

We note that if  $f(0) = 0$  then (1.7) in F) implies (4.4) for some suitable constant  $C_f$ .

Our first result in this section shows that global existence of (4.1) can be established under much weaker assumption on the  $L^p$  norm of strong solutions to the following systems.

$$\begin{cases} u_t = \operatorname{div}(A(\sigma u)Du) + \hat{f}(\sigma u, \sigma Du) & (x, t) \in Q = \Omega \times (0, T_0), \sigma \in [0, 1] \\ u(x, 0) = U_0(x) & x \in \Omega \\ u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T_0). \end{cases} \quad (4.5)$$

**Theorem 4.1** *Assume  $SKT$ ),  $F'$ ). Let  $u_\sigma$  be a strong solution to (4.5) and  $(0, T_0)$  be its maximal existence interval. Suppose that there are positive constants  $q > 1$  and  $M_{U_0, T_0}$  depending on  $\|U_0\|_{W^{1,p_0}(\Omega)}$  and  $T_0$  such that*

$$\|u_\sigma(\cdot, t)\|_{L^{qk}(\Omega)} \leq M_{U_0, T_0} \quad \text{for all } t \in (0, T_0). \quad (4.6)$$

Then

$$\int_{\Omega \times \{t\}} \lambda(u_\sigma) |Du_\sigma|^2 dx \leq C(T_0, C_f, M_{U_0, T_0}) \lambda_S \lambda_0^{-1} \quad \forall t \in (0, T_0). \quad (4.7)$$

Moreover, (4.1) or (4.5) for  $\sigma = 1$  has a unique strong solution  $u$  which exists globally, i.e.  $T_0 = \infty$ .

Next, if we can control the bound (4.6) uniformly when  $t$  is large then we have the following result on the existence of global attractors.

**Theorem 4.2** *Assume  $SKT$ ),  $F'$ ). Suppose that there are constants  $q > 1$  and  $M$  such that the dynamical system defined by (4.1) possesses an absorbing ball in  $L^{qk}(\Omega)$ . That is there is a constant  $M$  such that for any bounded set  $K \subset W^{1,p_0}(\Omega)$  there is a  $T_K > 0$  and any global solution  $u$  of (4.1) with initial data in  $K$  will satisfy*

$$\|u(\cdot, t)\|_{L^{qk}(\Omega)} \leq M \quad \text{for all } t \geq T_K. \quad (4.8)$$

Then

$$\int_{\Omega \times \{t\}} \lambda(u) |Du|^2 dx \leq C(M) \lambda_S \lambda_0^{-1} \quad \text{for all } t \geq T_K + 1. \quad (4.9)$$

Moreover, the system (4.1) possesses a global attractor in  $X = W^{1,p_0}(\Omega)$ .

The proof of the above theorems will be based on the checking of the assumptions of on the boundedness of  $\|u\|_{W^{1,2}(\Omega)}$  and  $\|\lambda^{s_0}(u)\|_{L^1(\Omega)}$  in Theorem 2.1 and Theorem 2.3. By Sobolev's embedding theorems for  $n = 2$  and the polynomial growth assumption on  $\lambda(u)$ , we need only establish the corresponding boundedness of  $\|u\|_{W^{1,2}(\Omega)}$ . This is clear from (4.7), (4.9) and the fact that  $\lambda(u)$  is bounded from below.

In the sequel, when there is no ambiguity  $C, C_i$  will denote universal constants that can change from line to line in our argument. If necessary,  $C(\cdots)$  is used to denote quantities which are bounded in terms of their parameters. Furthermore, as we will always consider a strong solution to (4.1) that exists in its maximal time interval  $(0, T_0)$ , the derivatives  $u_t, D^2u$  and  $Du_t$  make sense in the proof below.

We first have the following lemma establishing a differential inequality which will be used in several places later on.

**Lemma 4.3** Assume  $A, F$  and that  $A(u) = P_u(u)$  for some  $P : \mathbb{R}^m \rightarrow \mathbb{R}^m$ . Let  $u$  be a strong solution to (4.1) on some interval  $(0, T_0)$ . For any nonnegative  $C^1$  function  $\eta(t)$  on  $[0, \infty)$  we have

$$\begin{aligned} \int_{\Omega} \lambda(u) |u_t|^2 \eta^2 dx + \frac{d}{dt} \int_{\Omega} |A(u) Du|^2 \eta^2 dx \leq \\ C \int_{\Omega} [|A(u) Du|^2 (\eta \eta_t + 1) + \lambda(u) |f(u)|^2 \eta^2] dx. \end{aligned} \quad (4.10)$$

**Proof:** We test the system of  $u$  by  $A(u) u_t \eta^2(t)$  (i.e. multiplying the  $i^{th}$  equation of (4.1) by  $\sum_j a_{ij}(u) (u_j)_t \eta^2$ ,  $A(u) = (a_{ij}(u))$ , integrating over  $\Omega$  and summing the results) and integrate by parts in  $x$  to get

$$\int_{\Omega} (\langle A(u) u_t, u_t \rangle + \langle A(u) Du, D(A(u) u_t) \rangle) \eta^2 dx = \int_{\Omega} \langle \hat{f}(u, Du), A(u) u_t \rangle \eta^2 dx.$$

As  $D(A(u) u_t) = D(P(u)_t) = (DP(u))_t = (A(u) Du)_t$ , we see that

$$\frac{1}{2} \frac{\partial}{\partial t} (|ADu|^2 \eta^2) = \langle A(u) Du, D(A(u) u_t) \rangle \eta^2 + |A(u) Du|^2 \eta \eta_t,$$

and obtain

$$\begin{aligned} \int_{\Omega} [\langle A(u) u_t, u_t \rangle \eta^2 + \frac{1}{2} \frac{\partial}{\partial t} (|ADu|^2 \eta^2)] dx = \\ \int_{\Omega} [|A(u) Du|^2 \eta \eta_t + \langle \hat{f}(u, Du), A(u) u_t \rangle \eta^2] dx. \end{aligned} \quad (4.11)$$

We now use the ellipticity of  $A(u)$  in the first integrand on the left hand side to have  $\langle A(u) u_t, u_t \rangle \geq \lambda(u) |u_t|^2$ . Also, as  $|\hat{f}(u, Du)| \leq C \lambda^{\frac{1}{2}}(u) |Du| + f(u)$  and  $|A(u)| \leq C \lambda(u)$ , we use Young's inequality to find a constant  $C(\varepsilon)$  such that for any  $\varepsilon > 0$  we can estimate the second integrand on the right hand side as follows

$$|\langle \hat{f}(u, Du), A(u) u_t \rangle| \leq \varepsilon \lambda(u) |u_t|^2 + C(\varepsilon) [\lambda^2(u) |Du|^2 + \lambda(u) |f(u)|^2].$$

Hence, using this in (4.11) with sufficiently small  $\varepsilon$  and noting that  $|A(u) Du|^2 \sim \lambda^2(u) |Du|^2$ , see (4.3), we get (4.10). ■

The integrand  $\lambda(u) |f(u)|^2 \eta^2$  on the right hand side of (4.10) will play an important role in our analysis so that the following lemma will show that it can be controlled under some boundedness assumption on the  $L^p$  norm of  $u$ .

To proceed we collect some well known inequalities here for later use. For any  $p \geq 1$ ,  $\alpha \in (0, 1)$  and  $\varepsilon > 0$  we have the following inequality for all  $w \in W^{1,2}(\Omega)$ , recalling that  $n = 2$

$$\left( \int_{\Omega} w^p dx \right)^{\frac{2}{p}} \leq \varepsilon \int_{\Omega} |Dw|^2 dx + C(\varepsilon) \left( \int_{\Omega} w^\alpha dx \right)^{\frac{2}{\alpha}}. \quad (4.12)$$

Concerning the last integral, for  $w = \lambda_0 |u|^p + \lambda_1 |u|^q$  and sufficiently small  $\alpha$  we note the following simple fact which results from Hölder's inequality

$$\left( \int_{\Omega} (\lambda_0 |u|^p + \lambda_1 |u|^q)^\alpha dx \right)^{\frac{2}{\alpha}} \leq C(\alpha) (\lambda_0^2 \|u\|_{L^1(\Omega)}^{2p} + \lambda_1^2 \|u\|_{L^1(\Omega)}^{2q}). \quad (4.13)$$

Combining the above two inequalities, for  $w = |u|^{\frac{q}{2}+1}$ ,  $\lambda_0 = \lambda_1 = 1$  and  $p = 2$ , we have

$$\int_{\Omega} |u|^{q+2} dx \leq \varepsilon \int_{\Omega} |u|^q |Du|^2 dx + C(\varepsilon, q) \|u\|_{L^1(\Omega)}^{q+2}. \quad (4.14)$$

Similarly, letting  $w = \lambda(u)|u|$  and noting that  $w \sim |u|^{k+1}$  and  $|Dw| \sim \lambda(u)|Du|$ , for any  $p \geq 1$  we can find a constant  $C_p(\varepsilon, \|u\|_{L^1(\Omega)})$  such that (with  $\lambda_S = \lambda_0 + \lambda_1$ )

$$\left( \int_{\Omega} (\lambda(u)|u|)^p dx \right)^{\frac{2}{p}} \leq \varepsilon \int_{\Omega} \lambda^2(u) |Du|^2 dx + C_p(\varepsilon, \|u\|_{L^1(\Omega)}) \lambda_S^2. \quad (4.15)$$

**Lemma 4.4** *For some  $q > 1$  and  $t \in (0, T_0)$  we suppose that the number  $M(t) = \|u(\cdot, t)\|_{L^{qk}(\Omega)}$  is finite. Then, for any  $\varepsilon_0 > 0$  we can find a positive constant  $C(\varepsilon_0, M(t))$  such that*

$$\int_{\Omega \times \{t\}} \lambda(u) |f(u)|^2 dx \leq C_f^2 \lambda_S^{-1} \left[ \varepsilon_0 \int_{\Omega \times \{t\}} |A(u) Du|^2 dx + C(\varepsilon_0, M(t)) \lambda_S^2 \right]. \quad (4.16)$$

**Proof:** We write

$$\lambda(u) |f(u)|^2 = \lambda^2(u) |u|^2 \Lambda(u), \text{ where } \Lambda(u) = \frac{|f(u)|^2}{|u|^2 \lambda(u)}. \quad (4.17)$$

For any  $q > 1$ , and  $q' = q/(q-1)$  we can use Hölder's inequality to have

$$\int_{\Omega} \lambda(u) |f(u)|^2 dx = \int_{\Omega} \lambda^2(u) |u|^2 \Lambda(u) dx \leq C \left( \int_{\Omega} (\lambda(u) |u|)^{2q'} dx \right)^{\frac{1}{q'}} \left( \int_{\Omega} \Lambda^q(u) dx \right)^{\frac{1}{q}}.$$

For the first factor on the right we use (4.15) with  $w = \lambda(u)|u|$  with  $p = 2q'$  and the fact that  $\|u(\cdot, t)\|_{L^1(\Omega)}$  is bounded by  $M(t) = \|u(\cdot, t)\|_{L^{qk}(\Omega)}$  to find a constant  $C_0(\varepsilon, M(t))$  such that

$$\left( \int_{\Omega} (\lambda(u) |u|)^{2q'} dx \right)^{\frac{1}{q'}} \leq \varepsilon \int_{\Omega} \lambda^2(u) |Du|^2 dx + C_0(\varepsilon, M(t)) \lambda_S^2.$$

On the other hand, from the assumption (4.4) in F'), for some constant  $C$  we have  $\Lambda(u) \leq C_f^2 \lambda_S^{-2} \lambda(u)$  so that, using the growth condition on  $\lambda(u)$

$$\left( \int_{\Omega} \Lambda^q(u) dx \right)^{\frac{1}{q}} \leq C_f^2 \lambda_S^{-2} \|\lambda(u)\|_{L^q(\Omega)} \leq C_f^2 \lambda_S^{-1} (1 + \|u\|_{L^{kq}(\Omega)}) \leq C_f^2 C_1(M(t)) \lambda_S^{-1}.$$

Therefore, for any given  $\varepsilon_0 > 0$  we choose  $\varepsilon = \varepsilon_0 C_1^{-1}(M(t))$  and combine the above estimates to obtain a constant  $C(\varepsilon_0, M(t))$  such that

$$\int_{\Omega} \lambda(u) |f(u)|^2 dx \leq C_f^2 \lambda_S^{-1} \left[ \varepsilon_0 \int_{\Omega} \lambda^2(u) |Du|^2 dx + C(\varepsilon_0, M(t)) \lambda_S^2 \right].$$

By (4.3),  $|A(u) Du|^2 \sim \lambda^2(u) |Du|^2$ , the above proves the lemma. ■

We now apply Theorem 2.1 to establish the global existence result.

**Proof of Theorem 4.1:** From (4.10) of Lemma 4.3 with  $\eta \equiv 1$ , we see that

$$\frac{d}{dt} \int_{\Omega} |ADu|^2 dx \leq C \int_{\Omega} (|A(u)Du|^2 + \lambda(u)|f(u)|^2) dx. \quad (4.18)$$

We let  $\varepsilon_0 = 1$  in (4.16) and replace  $M(t)$  by  $M_{U_0, T_0}$ , see (4.6), to obtain a constant  $C(M_{U_0, T_0})$  such that

$$\int_{\Omega} \lambda(u)|f(u)|^2 dx \leq C_f^2 \lambda_S^{-1} \left[ \int_{\Omega} |A(u)Du|^2 dx + C(M_{U_0, T_0}) \lambda_S^2 \right].$$

We then have for  $\beta := C_f^2 \lambda_S^{-1}$  and some constant  $C$  the following inequality.

$$\frac{d}{dt} \int_{\Omega} |ADu|^2 dx \leq C(\beta + 1) \int_{\Omega} |A(u)Du|^2 dx + C_f^2 C(M_{U_0, T_0}) \lambda_S. \quad (4.19)$$

We now set

$$y(t) = \int_{\Omega} |A(u)Du(x, t)|^2 dx$$

to see from (4.19) that  $y'(t) \leq C(\beta + 1)y(t) + C(M_{U_0, T_0}, C_f)\lambda_S$  for all  $t \in (0, T_0)$ . A simple use of Gronwall's inequality shows that there exists a constant  $C(T_0, C_f, M_{U_0, T_0})$  such that

$$\int_{\Omega \times \{t\}} |A(u)Du|^2 dx \leq C(T_0, C_f, M_{U_0, T_0}) \lambda_S \quad \forall t \in (0, T_0). \quad (4.20)$$

Because  $|A(u)Du|^2 \geq C\lambda^2(u)|Du|^2$  for some  $C > 0$  and  $\lambda(u)$  is bounded from below by  $\lambda_0$ , the above yields the bound (4.7) and a bound for  $\|Du(\cdot, t)\|_{L^2(\Omega)}$  for all  $t > 0$ . We see that Theorem 2.1 applies here. In fact, it is easy to check that the data of the systems in (2.1) satisfies SKT) and F') with the same set of constants so that the above argument (for  $\sigma = 1$ ) yields a uniform estimate for  $\|u_{\sigma}(\cdot, t)\|_{W^{1,2}(\Omega)}$  in (2.3) of Theorem 2.1. The bound for  $\|\lambda^{s_0}(u_{\sigma}(\cdot, t))\|_{L^1(\Omega)}$  in (2.4) follows from the polynomial growth of  $\lambda(u)$  and Sobolev's inequality. The proof is then complete. ■

We now turn to the proof of Theorem 4.2 and apply Theorem 2.3. We need only show that there is an absorbing ball in  $W^{1,2}(\Omega)$ . To proceed we need some lemmas establishing uniform estimates for  $\|Du\|_{L^2(\Omega)}$  under much weaker assumptions on the  $L^p$  norms of  $u$ .

**Lemma 4.5** *Let  $u$  be a strong solution to (4.1) on some interval  $(0, T_0)$ . For any  $\tau_0 \in (0, 1)$  assume that there are positive constants  $q > 1$ ,  $T \geq 0$  and  $T' = \min\{T + \tau_0, T_0\}$  such that the number*

$$M_{T, T', q} = \sup_{t \in [T, T']} \|u(\cdot, t)\|_{L^{qk}(\Omega)} \quad (4.21)$$

*is finite. Then there exists a positive constant  $C_0(M_{T, T', q}, C_f)$  such that*

$$\int_{T+\tau_0}^{T'} \int_{\Omega} |A(u)Du|^2 dx ds \leq C_0(M_{T, T', q}, C_f)(T' - T)\left(\frac{1}{\tau_0} + \lambda_S\right). \quad (4.22)$$

**Proof:** For  $T \geq 0$ ,  $\tau_0 > 0$  and  $T' = \min\{T + \tau_0, T_0\}$  and any  $R \in (0, \tau_0)$  let us denote

$$I_R = [T + \tau_0 - R, T'].$$

Also, for any  $0 \leq \rho < R \leq \tau_0$  we let  $\eta$  be a cut-off function for  $I_R, I_\rho$ . That is,  $\eta \equiv 1$  in  $I_\rho$  and  $\eta(\tau) \equiv 0$  for  $\tau \leq T + \tau_0 - R$  and  $|\eta_t| \leq 1/(R - \rho)$ .

We multiply the system (4.1) by  $P(u)\eta^2(t)$  and integrate by parts in  $x$  to get

$$\int_{\Omega} \langle A(u)Du, A(u)Du \rangle \eta^2 dx = \int_{\Omega} (-\langle P(u), u_t \rangle + \langle \hat{f}(u, Du), P(u) \rangle) \eta^2 dx.$$

Since  $|\hat{f}(u, Du)| \leq C\lambda^{\frac{1}{2}}(u)|Du| + f(u)$  and  $|A(u)Du|^2 \sim \lambda^2(u)|Du|^2$  (see (4.3)), we apply Young's inequality to the first term in the integrand on the right hand side and find a constant  $C_1$  such that for any  $\varepsilon_* > 0$

$$\int_{\Omega} |A(u)Du|^2 \eta^2 dx \leq C_1 \int_{\Omega} (\varepsilon_* \lambda(u) |u_t|^2 + \varepsilon_*^{-1} \lambda^{-1}(u) |P(u)|^2 + |f(u)| |P(u)|) \eta^2 dx.$$

Integrate the above over  $I_R$  to get

$$\begin{aligned} \int_{I_R} \int_{\Omega} |A(u)Du|^2 \eta^2 dx ds &\leq \\ C_1 \int_{I_R} \int_{\Omega} (\varepsilon_* \lambda(u) |u_t|^2 + \varepsilon_*^{-1} \lambda^{-1}(u) |P(u)|^2 + |f(u)| |P(u)|) \eta^2 dx ds. \end{aligned} \quad (4.23)$$

We integrate (4.10) of Lemma 4.3 over  $I_R$  and note that  $|ADu|^2 \eta^2 = 0$  at  $T + \tau_0 - R$ . From the choice of  $\eta$  we can find a constant  $C_2$  to obtain

$$\begin{aligned} \int_{I_R} \int_{\Omega} \lambda(u) |u_t|^2 \eta^2 dx ds + \int_{\Omega \times \{T'\}} |A(u)Du|^2 dx &\leq \\ C_2 \int_{I_R} \int_{\Omega} [(\frac{1}{R - \rho} + 1) |A(u)Du|^2 + \lambda(u) |f(u)|^2 \eta^2] dx ds. \end{aligned}$$

Hence,

$$\int_{I_R} \int_{\Omega} \lambda(u) |u_t|^2 \eta^2 dx ds \leq C_2 \int_{I_R} \int_{\Omega} [(\frac{1}{R - \rho} + 1) |A(u)Du|^2 + \lambda(u) |f(u)|^2 \eta^2] dx ds. \quad (4.24)$$

We now take  $\varepsilon_* = \frac{1}{2}(C_1 C_2)^{-1}(R - \rho)$  in (4.23) and note that  $\varepsilon_* \leq C\tau_0$  for some fixed constant  $C$ . Hence, multiplying (4.24) by  $C_1 \varepsilon_*$  and using the result in (4.23) we easily get (using  $I_\rho \subset I_R$ )

$$\begin{aligned} \int_{I_\rho} \int_{\Omega} |A(u)Du|^2 dx ds &\leq \frac{1}{2}(1 + R - \rho) \int_{I_R} \int_{\Omega} |A(u)Du|^2 dx ds + \\ C_3 \int_{I_R} \int_{\Omega} \left( \frac{1}{|R - \rho|} \lambda^{-1}(u) |P(u)|^2 + |f(u)| |P(u)| + \lambda(u) |f(u)|^2 \right) dx ds. \end{aligned} \quad (4.25)$$

We then estimate the last integral on the right hand side of (4.25). In the sequel, we will abbreviate

$$M := M_{T, T', q} \text{ and } |I| = T' - T.$$

Firstly, from SKT) we have  $\lambda^{-1}(u)|P(u)|^2 \leq C\lambda(u)|u|^2$ . The integral of  $\lambda(u)|u|^2$  can be estimated by using (4.12) for  $w = \lambda^{\frac{1}{2}}(u)u$  and the fact that  $\|u(\cdot, t)\|_{L^1(\Omega)}$  is bounded in terms of  $M$  for  $t \in (T, T')$ . We then have

$$\begin{aligned} \int_{I_R} \int_{\Omega} \lambda^{-1}(u)|P(u)|^2 dx ds &\leq C \int_{I_R} \int_{\Omega} (\lambda(u)|Du|^2 + C_4(M)\lambda_S) dx ds \\ &\leq C_5(M)|I|(1 + \lambda_S). \end{aligned}$$

Here, we have used the fact that there is a constant  $C(M, \tau_0)$  such that

$$\int_{I_R} \int_{\Omega} \lambda(u)|Du|^2 dx \leq C(M, \tau_0)|I|. \quad (4.26)$$

This can be proved easily by testing the system with  $u$  (see also Remark 4.9 for details).

Next, using (4.12) for  $w = P(u)$  and any  $\varepsilon_0 > 0$  we can find  $C(\varepsilon_0, M)$  such that

$$\int_{I_R} \int_{\Omega} |P(u)|^2 dx ds \leq \int_{I_R} \int_{\Omega} (\varepsilon_0|A(u)Du|^2 + C(\varepsilon_0, M)\lambda_S^2) dx ds. \quad (4.27)$$

By F'),  $|f(u)| \leq C_f \lambda_S^{-1}|u|\lambda(u)$ . Applying Young's inequality, we have

$$|f(u)||P(u)| \leq C_f^{-1}\lambda_S|f(u)|^2 + C_f\lambda_S^{-1}|P(u)|^2 \leq C_f\lambda_S^{-1}(|u|^2|\lambda(u)|^2 + |P(u)|^2).$$

Using (4.15) ( $p = 2$ ) and (4.27) to estimate the integral of the right hand side of the above, we obtain

$$\int_{I_R} \int_{\Omega} |f(u)||P(u)| dx ds \leq C_f\lambda_S^{-1} \int_{I_R} \int_{\Omega} (\varepsilon_0|A(u)Du|^2 + C(\varepsilon_0, M)\lambda_S^2) dx ds.$$

Concerning the last integrand on the right hand side of (4.25), by (4.16), we have  $|f(u)| \leq C_f\lambda_S^{-1}|u|\lambda(u)$

$$\int_{\Omega} \lambda(u)|f(u)|^2 dx \leq C_f^2\lambda_S^{-1} \left[ \varepsilon_0 \int_{\Omega} |A(u)Du|^2 dx + C(\varepsilon_0, M)\lambda_S^2 \right].$$

We now set for  $\tau \in [0, \tau_0]$

$$F(\tau) := \int_{I_\tau} \int_{\Omega} |A(u)Du|^2 dx ds$$

and use the above estimates for the integrals on the right hand side of (4.25) to obtain

$$F(\rho) \leq \mu F(R) + \frac{\lambda_S C_6(M)}{R - \rho} |I| + C(\varepsilon_0, M, C_f) |I|(1 + \lambda_S), \quad (4.28)$$

where for some constant  $C_7$

$$\mu = \frac{1}{2}(1 + R - \rho) + C_7(C_f + C_f^2)\varepsilon_0\lambda_S^{-1}. \quad (4.29)$$

Since  $\lambda_S$  is bounded from below and  $R - \rho \leq \tau_0 < 1$ , we can choose  $\varepsilon_0$  sufficiently small and depending only on  $C_f$  such that  $\mu < 1$ . By an elementary iteration lemma [7, Lemma 6.1, p.192], we have

$$F(\rho) \leq C \left[ \frac{C_6(M)\lambda_S}{R - \rho} |I| + C_8(M, C_f) |I|\lambda_S \right] \text{ for all } 0 \leq \rho < R \leq \tau_0.$$



We now take  $R = \tau_0$ ,  $\rho = 0$  in the above inequality to obtain

$$\int_{T+\tau_0}^{T'} \int_{\Omega} |A(u)Du|^2 dx ds \leq C(M, C_f) |I| \lambda_S \left( \frac{1}{\tau_0} + 1 \right).$$

The above estimate completes the proof as  $M := M_{T, T', q}$  and  $|I| = T' - T$ . ■

We are now ready to give

**The proof of Theorem 4.2:** We will apply Theorem 2.3 here. To this end, let  $K$  be any bounded subset of  $X$  and  $u$  be the solution of (4.1) with initial data  $U_0$  in  $K$ . Under the assumption (4.8)

$$\|u(\cdot, t)\|_{L^{qk}(\Omega)} \leq M \quad \forall t \geq T_K, \quad (4.30)$$

we will show that there exists a constant  $C(M)$  such that

$$\int_{\Omega \times \{t\}} \lambda^2(u) |Du|^2 dx \leq C(M) \lambda_S \quad \forall t \geq T_K + 1. \quad (4.31)$$

Because  $\lambda(u)$  is bounded from below by  $\lambda_0$ , (4.31) yields a uniform bound for  $\|Du(\cdot, t)\|_{L^2(\Omega)}$ ,  $t > T_K + 1$ . Theorem 2.3 then applies and completes the proof.

First of all, by the global existence result we can assume  $T_K > 1$  and take  $\tau_0 = 1/2$  in Lemma 4.5. From (4.16) with  $\varepsilon_0 = 1$  and suitable choice of  $C(M)$ , we have

$$\int_{\Omega} \lambda(u) |f(u)|^2 dx \leq C_f^2 \lambda_S^{-1} \left[ \int_{\Omega} |A(u)Du|^2 dx + C(M) \lambda_S^2 \right] \quad \forall t \geq T_K.$$

As in the proof of Theorem 4.1, see (4.19), we obtain for  $\beta := C_f^2 \lambda_S^{-1}$  and  $\alpha := C_f^2 C(M)$

$$\frac{d}{dt} \int_{\Omega} |A(u)Du|^2 dx \leq C(\beta + 1) \int_{\Omega} |A(u)Du|^2 dx + \alpha \lambda_S \quad \forall t \geq T_K. \quad (4.32)$$

We now set

$$y(t) = \lambda_S^{-1} \int_{\Omega} |A(u)Du(x, t)|^2 dx$$

and derive from (4.32) that

$$y'(t) \leq C(\beta + 1)y(t) + \alpha \quad \forall t \in (T_K, \infty).$$

For any  $\tau_0 > 0$  the uniform Gronwall inequality (see [18, Lemma 1.1, p.91]) then gives

$$y(t + \tau_0) \leq \left[ \frac{a_3}{\tau_0} + a_2 \right] \exp(a_1) \quad \forall t > T_K, \quad (4.33)$$

where

$$a_1 := \int_t^{t+\tau_0} C(\beta + 1) ds, \quad a_2 := \int_t^{t+\tau_0} \alpha ds, \quad a_3 := \int_t^{t+\tau_0} y(s) ds.$$

From the definitions of  $\alpha, \beta$

$$a_1 \leq C(C(M) C_f^2 \lambda_S^{-1} + 1) \tau_0 \quad \text{and} \quad a_2 \leq C(M) C_f^2 \tau_0.$$

For a fixed  $\tau_0 \in (0, 1)$  and any  $t \geq T_K + \tau_0$  we take  $T$  and  $T'$  in (4.22) such that  $T + \tau_0 = t$  and  $T' = t + \tau_0$ . Then  $T \geq T_K$  so that Lemma 4.5 applies here to give

$$\frac{a_3}{\tau_0} = (\tau_0 \lambda_S)^{-1} \int_t^{t+\tau_0} \int_{\Omega} |A(u) Du(x, t)|^2 dx ds \leq \lambda_S^{-1} C_0(M, C_f) \left( \frac{1}{\tau_0} + \lambda_S \right).$$

Putting these estimates in (4.33) and the fact that  $\lambda_S$  is bounded from below, we see that  $y(t + \tau_0)$  is uniformly bounded by a constant  $C(M)$  for  $t > T_K + \tau_0$ . Using the definition of  $y(t)$  and letting  $\tau_0 = 1/2$  we obtain (4.31) and complete the proof. ■

Next, we will show that the assumption on the boundedness of  $L^p$  norm of the solutions in (4.6) and (4.8) can be weaken further if a mild hypothesis on the structure on  $A(u)$  is imposed. In fact, the  $L^p$  norm can be replaced by  $L^1$  one if we assume further the following (the constant  $C_*$  is the ratio between the eigenvalues of  $A(u)$  described in A)).

**SG')** If  $k > 2$  then there is a number  $\delta_k \in (0, 1)$  such that  $(k - 2)/k \leq \delta_k C_*^{-1}$ .

**Theorem 4.6** Assume  $SKT)$ ,  $F')$  and  $SG')$ . The conclusions of Theorem 4.1 and Theorem 4.2 hold if (4.6) and (4.8) respectively hold for the  $L^1(\Omega)$  norm of  $u$ .

The proof of this theorem clearly follows from the following lemma (and Remark 4.8) which shows that an appropriate bound for the  $L^1$  norm of  $u$  implies those of  $L^{qk}$  norm of  $u$  for some  $q > 1$ . Theorem 4.1 and Theorem 4.2 then apply.

**Lemma 4.7** Assume  $SKG)$ ,  $F')$  and  $SG')$ . Suppose that there are  $T_* \in (0, T_0)$  and a continuous function  $C_0$  on  $(T_*, \infty)$  such that

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq C_0(t) \quad \text{for all } t \in (T_*, T_0). \quad (4.34)$$

Then, for any positive  $\tau_0 < T_0 - T_*$  there is a number  $q > 1$  such that

$$\int_{\Omega} |u(x, t)|^{qk} dx \leq C \left( \sup_{(T_*, T_0)} C_0(t), \tau_0 \right) \quad \forall t \in (T_* + \tau_0, T_0). \quad (4.35)$$

**Proof:** First of all, we recall the following fact from [8] if  $l > 0$  and

$$\frac{l}{l+2} \leq \delta_l C_*^{-1} \text{ for some } \delta_l \in (0, 1)$$

then there exists a positive  $\lambda_l$ , which depends on  $k$ , such that

$$\langle A(u) Du, D(|u|^l u) \rangle \geq \lambda_l \lambda(u) |u|^l |Du|^2. \quad (4.36)$$

By SG') it is clear that we can find  $l > \max\{0, k - 2\}$  such that (4.36) holds.

Let  $T \geq T_*$  such that there is  $T' \in (T + \tau_0, T_0)$ . We test the system of  $u$  with  $|u|^l u \eta^p(t)$ , where  $\eta$  is a cutoff function for  $[T, T']$ ,  $[T + \tau_0, T']$  and  $p > 1$ , which will be determined shortly. Using (4.36) and the fact that  $|\eta_t| \leq 1/\tau_0$ , we easily obtain for  $Q = \Omega \times [T, T']$

$$\begin{aligned} \sup_{t \in [T + \tau_0, T']} \int_{\Omega} |u|^{l+2} dx + \lambda_l \iint_Q \lambda(u) |u|^l |Du|^2 \eta^p dz \leq \\ C \iint_Q [\hat{f}(u, Du), |u|^l u \eta^p + \frac{1}{\tau_0} |u|^{l+2} \eta^{p-1}] dz. \end{aligned} \quad (4.37)$$

For the last term in the integrant on the right, we can choose  $p$  such that  $p-1 > p \frac{l+2}{k+l+2}$  and use Young's inequality to find some positive constant  $C(k, \tau_0)$  such that

$$\frac{1}{\tau_0} |u|^{l+2} \eta^{p-1} \leq |u|^{k+l+2} \eta^p + C(k, \tau_0) \leq |u|^{k+l+2} \eta^p + C(k, \tau_0).$$

By SKT) and F'), we can use Young's inequality to find a constant  $C$  such that

$$\langle \hat{f}(u, Du), |u|^l u \rangle \leq \varepsilon \lambda(u) |u|^l |Du|^2 + C(\varepsilon) (|u|^{k+l+2} + 1).$$

Using (4.14) for  $q = k + l$  and the assumption (4.34), we have for  $t \in (T_*, T')$

$$\begin{aligned} \int_{\Omega \times \{t\}} |u|^{k+l+2} \eta^p dx &\leq \varepsilon \int_{\Omega \times \{t\}} |u|^{k+l} |Du|^2 \eta^p dx + C(\varepsilon, C_0(t)) \\ &\leq \varepsilon \int_{\Omega \times \{t\}} |u|^l \lambda(u) |Du|^2 \eta^p dx + C(\varepsilon, C_0(t)). \end{aligned}$$

Thus, there is a constant  $C(\varepsilon, \sup_{(T, T')} C_0(t), k, \tau_0)$  such that the right hand side of (4.37) can be estimated by

$$\varepsilon \iint_Q |u|^l \lambda(u) |Du|^2 \eta^p dz + C(\varepsilon, \sup_{(T, T')} C_0(t), k, \tau_0) |\Omega| (T' - T).$$

Choosing  $\varepsilon = \lambda_l/2$ , we can deduce from (4.37) and the above inequality the following estimate.

$$\sup_{t \in [T+\tau_0, T']} \int_{\Omega} |u|^{l+2} dx + \iint_Q |u|^l \lambda(u) |Du|^2 \eta^p dz \leq C(\sup_{(T, T')} C_0(t), k, \tau_0) |\Omega| (T' - T). \quad (4.38)$$

Therefore,

$$\int_{\Omega} |u(x, t)|^{l+2} dx \leq C(\sup_{(T_*, T')} C_0(t), k, \tau_0, T_0) |\Omega| \quad \forall t \geq T + \tau_0.$$

Since  $l + 2 > k$ , there is  $q > 1$  such that  $l + 2 = qk$  and the above yields (4.35). The proof is then complete. ■

**Remark 4.8** We can allow  $T_* = 0$  and  $\tau_0 = 0$  by letting  $\eta \equiv 1$ . In this case, (4.37) now is

$$\begin{aligned} \sup_{t \in [0, T']} \int_{\Omega} |u|^{l+2} dx + \lambda_l \iint_Q \lambda(u) |u|^l |Du|^2 dz &\leq \\ &C \iint_Q \langle \hat{f}(u, Du), |u|^l u \rangle dz + \int_{\Omega} |u(x, 0)|^{l+2} dx. \end{aligned}$$

We can see that the proof can continue to give (4.35) with the right hand side depending on  $\|u(\cdot, 0)\|_{L^{l+2}(\Omega)}$ . This suffices to give the global existence result of Theorem 4.1. Once this is established, we can take  $\tau_0 = 1$  to get a uniform bound for  $\|u(\cdot, t)\|_{L^{l+2}(\Omega)}$ , independent of  $\|u(\cdot, 0)\|_{L^{l+2}(\Omega)}$ , and Theorem 4.2 can apply.

**Remark 4.9** By the ellipticity of  $A(u)$  (4.36) holds if  $l = 0$ , i.e. we test the system with  $u$ , and we do not need SG') here. In this case, (4.38) provides the estimate

$$\int_{T+\tau_0}^{T'} \int_{\Omega} \lambda(u) |Du|^2 \eta^p dx dx dt \leq C \left( \sup_{(\tau_0, T')} C_0(t), \tau_0 \right) |\Omega| (T' - T).$$

This is (4.26) which was used in the proof of Lemma 4.5.

We conclude this paper by giving a simple example proving the existence of a global attractor of the generalized version (1.1). Inspired by the *competitive* Lotka-Volterra reaction in (1.1), we assume that  $\hat{f}(u, Du)$  is of the form

$$\hat{f}(u, Du) = B(u)Du + Ku - G(u)u, \quad (4.39)$$

where  $K, B(u), G(u)$  are  $m \times m$  matrices and  $K$  is a constant one.

**Theorem 4.10** Assume SKT), (4.39) and that  $B(u), G(u)$  are  $C^1$  in  $u$ . We assume that  $|B(u)| \leq C\lambda^{\frac{1}{2}}(u)$  for all  $u \in \mathbb{R}^m$ . In addition,  $G(u)$  is positive definite in the following sense: there are  $c_0 > 0$  and  $\kappa \in (0, k]$  such that for all  $w, u \in \mathbb{R}^m$

$$\langle G(w)u, u \rangle \geq c_0 |w|^\kappa |u|^2, \quad |G(u)| \sim |u|^\kappa, \quad |G_u(u)| \sim |u|^{\kappa-1}. \quad (4.40)$$

Then (4.1) defines a dynamical system which possesses a global attractor on  $X = W^{1,p_0}(\Omega)$ ,  $p_0 > 2$ .

**Proof:** We first consider the global existence by applying Theorem 4.1 (and then Theorem 4.6). By Remark 2.2, we can replace  $\hat{f}(\sigma u, \sigma Du)$  in (4.5) by

$$\hat{f}_\sigma(u, Du) = \sigma^k B(\sigma u)Du + f_\sigma(u), \quad f_\sigma(u) := \sigma^k Ku - \sigma^{k-\kappa} G(\sigma u)u \quad \sigma \in [0, 1].$$

Since  $|\partial_u f_\sigma(u)| \leq \sigma^k |K| + \sigma^{k-\kappa} |G(\sigma u)| + \sigma^{k-\kappa+1} |u| |\partial_{\sigma u} G(\sigma u)| \leq C\lambda(\sigma u)$  by the last two conditions in (4.40) and the fact that  $\lambda(\sigma u) \sim \lambda_0 + \sigma^k |u|^k$ , we see that  $f_\sigma$  satisfy (4.4) in F'). Hence, the argument in the proof of lemmas leading to Theorem 4.1 continues to hold if we can show that  $\|u_\sigma\|_{L^1(\Omega)}$  can be bounded uniformly for all solutions  $u_\sigma$  to (4.5) in any finite time intervals.

For any  $\sigma \in [0, 1]$ , let  $u$  be the solution of (4.5). From (4.40), with  $w = \sigma u$ , the bound on  $B(u)$  and Young's inequality, we have

$$\langle G(\sigma u)u, u \rangle \geq C_1 \sigma^\kappa |u|^{\kappa+2}, \quad \langle B(\sigma u)Du, u \rangle \leq \varepsilon \lambda(\sigma u) |Du|^2 + C(\varepsilon) |u|^2.$$

Multiplying (4.5) with  $u$ , integrating by parts in  $x$ , and using the above inequalities with sufficiently small  $\varepsilon$ , we easily obtain

$$\frac{d}{dt} \int_{\Omega} |u|^2 dx + \int_{\Omega} \lambda(\sigma u) |Du|^2 dx \leq C_1 \sigma^k \int_{\Omega} |u|^2 dx - C_2 \sigma^k \int_{\Omega} |u|^{\kappa+2} dx.$$

By Hölder's inequality, we can find a constant  $C_3 > 0$  such that

$$C_3 \left( \int_{\Omega} |u|^2 dx \right)^p \leq C_2 \int_{\Omega} |u|^{\kappa+2} dx, \quad p := (\kappa + 2)/2 > 1. \quad (4.41)$$

Therefore, for  $y(t) := \|u(\cdot, t)\|_{L^2(\Omega)}^2$

$$y'(t) \leq F(y(t)), \quad F(y) := \sigma^k(C_1 y - C_3 y^p).$$

Since  $p > 1$ , we see that  $F(y) \leq 0$  if  $y \geq y_* := (C_1/C_3)^{1/(p-1)}$ . As  $y(0) \geq 0$ , it follows that  $y(t) \leq \max\{y(0), y_*\}$ . Hence,  $\|u\|_{L^2(\Omega)}^2$  is bounded by a constant, independently of  $\sigma$ . This gives a uniform bound for solutions to (4.5). The global existence result then follows.

We now turn to the existence of global attractors. We test the system with  $u$  and obtain, denoting  $|K|$  the matrix norm of  $K$

$$\frac{d}{dt} \int_{\Omega} |u|^2 dx + \int_{\Omega} \lambda(u) |Du|^2 dx \leq \int_{\Omega} (C\lambda^{\frac{1}{2}}(u) |Du| |u| + |K| |u|^2 - c_0 |u|^{\kappa+2}) dx. \quad (4.42)$$

Using Young's inequality, for any  $\varepsilon > 0$  we can find a constant  $C(\varepsilon)$  such that

$$C\lambda^{\frac{1}{2}}(u) |Du| |u| \leq \varepsilon \lambda(u) |Du|^2 + C(\varepsilon) |u|^2, \quad |u|^2 \leq \varepsilon |u|^{\kappa+2} + C(\varepsilon).$$

Applying the above inequalities to the first and second integrands on the right hand side of (4.42), for sufficiently small  $\varepsilon$  we then deduce the following.

$$\frac{d}{dt} \int_{\Omega} |u|^2 dx + c_1 \int_{\Omega} |u|^{\kappa+2} dx \leq c_2.$$

Here,  $c_1, c_2$  are positive constants depending only on  $|K|, c_0$ . As in (4.41), we can apply Hölder's inequality to the second term on the left hand side to get another positive constant  $c_3$  which depend only on  $|K|, c_1, |\Omega|$  such that for  $y(t) := \|u(\cdot, t)\|_{L^2(\Omega)}^2$  and all  $t > 0$

$$y' + c_3 y^p \leq c_2, \quad p = (\kappa + 2)/2 > 1.$$

Using the uniform Gronwall's lemma ([18, Lemma 5.1]) with  $\gamma = c_3$  and  $\delta = c_2$ , we have

$$y(t) \leq (c_2/c_3)^{1/p} + (c_3(p-1)t)^{-1/(p-1)}. \quad (4.43)$$

For any fixed  $M_1 > (c_2/c_3)^{1/p}$  we let

$$T_* = \frac{1}{c_3(p-1)} \left( M_1 - (c_2/c_3)^{1/p} \right)^{1-p}.$$

It is easy to see from (4.43) that  $y(t) \leq M_1$  if  $t \geq T_*$ . The existence of the global attractor follows from this uniform estimate. ■

We remark that Theorem 4.10 applies to the system (1.1) with competitive Lotka-Volterra reaction terms and positive initial data  $U_0$ . However, the parameters  $A(u), f(u)$  of this system satisfy the assumption A),F) only for positive  $u$ . The positivity of solutions to (1.1) was established in [19] under suitable conditions on the parameter  $\alpha_{ij}$ 's. In a forthcoming paper [13] we will show that this is the case even in a much more general setting.

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